Asymptotic Analysis of the Abstract Telegraph Equation

Abstract

It is known that each solution \( u \) of the telegraph equation

\[
  u''(t) + 2au'(t) + A^2u(t) = 0
\]

is asymptotically equal to a solution \( v \) of the heat equation where \( A = A^* \) is a self-adjoint operator on a complex Hilbert space \( \mathcal{H} \) and \( a \) is a positive constant. The corresponding heat equation is

\[
  2av'(t) + A^2v(t) = 0.
\]

In other words,

\[
  \|u(t) - v(t)\| \leq \epsilon(t) \|v(t)\| \quad \text{as} \quad t \to \infty.
\]

Typically \( \|v(t)\| \to 0 \) and \( \epsilon(t) = Ce^{-\delta t} \) for some \( C \) and \( \delta \) depending on \( a, \ u(0), \ u'(0), \ A \).

We show how to find \( v(0) \) in terms of \( u(0) \) and \( u'(0) \) and \( a \), in a canonical way, so that one can say that a given solution \( u(t) \) of the telegraph equation is like a specific solution \( v(t) \) of the corresponding heat equation.

We explore this asymptotic relationship between the two equations in an unbounded domain with Wentzell boundary conditions.

This work is a collaborative effort with Gisèle Ruiz Goldstein, Jerome A. Goldstein, and Silvia Romanelli.